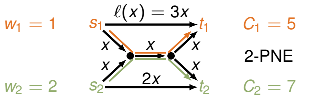
Existence and Complexity of Approximate Equilibria in Weighted Congestion Games

Weighted Congestion Games

- n players with weights w_i
- resources with latency functions $\ell_r(x)$ (polynomials of degree d, or general functions)
- strategies are subsets of resources

approximate PNE (α-PNE): no player can improve own cost unilaterally by more than a factor of α



Previous Results

- nonexistence of 1.153-PNE (Hansknecht et al.)
- existence of d-PNE (Caragiannis et al.)

For what values of α do α-PNE exist?

Complexity of deciding existence of α-PNE?

Our Results + Techniques

Improved gaps btw. nonexistence and existence

polynomial latencies: $\left(\Omega\left(\frac{\sqrt{d}}{\ln d}\right), d\right)$

general latencies: $\left(\Omega\left(\frac{n}{\ln n}\right), n\right)$

NP-completeness of deciding whether a game has an α-PNE for corresponding lower bounds

- *Construction* of a polynomial congestion game for any $d \geq 2$, with a super-constant number of players
- *Construction* of n-player congestion games with simple step functions having one breakpoint
- *Construction* of an unweighted polynomial congestion game from a Boolean circuit. Transfer hardness of CIRCUIT SAT to α-PNE
- *Combining* nonexistence and hardness gadgets to obtain a reduction from CIRCUIT SATISFIABILITY

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TUM

The Nonexistence Gadgets

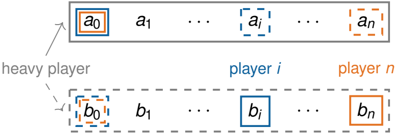
Polynomial Latencies

players:
a heavy player of weight 1,
n light players of weight $w \in [0, 1]$

resources: $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$

latencies: $k \in \{1, \ldots, d\}$ and $\beta \in [0, 1]$
for a_0 and b_0: $\ell_0(x) = x^k$
for others: $\ell_1(x) = \beta x^d$

strategies:
heavy player: $\{\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_n\}\}$
player i: $\{\{a_0, b_i\}, \{b_0, a_i\}\}$



General Latencies

players:
players $0, \ldots, m$
with weights $w_i = 1/2^i$

resources: $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$

latencies: (ξ positive solution of $(x + 1)^m = x^{m+1}$)

for a_0 and b_0: $\ell_0(x) = \begin{cases} 1, & \text{if } x \geq w_0 \\ 0, & \textit{otherwise} \end{cases}$

for a_i and b_i:

$$\ell_i(x) = \begin{cases} 1/\xi \, (1 + 1/\xi)^{i-1}, & \text{if } x \geq w_0 + w_i \\ 0, & \textit{otherwise} \end{cases}$$

strategies:
player 0: $\{\{a_0, \ldots, a_m\}, \{b_0, \ldots, b_m\}\}$
player i: $\{\{a_0, \ldots, a_{i-1}, b_i\}, \{b_0, \ldots, b_{i-1}, a_i\}\}$

Existence Result

Existence of n-PNE by showing that social cost is n-approximate potential for any congestion game.

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The Hardness Gadget

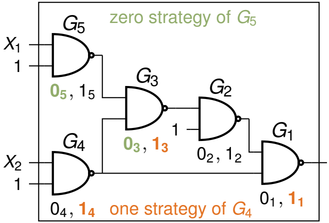
from a Boolean circuit using only 2-input NAND gates construct the following congestion game

players:
input players X_i,
gate players G_k (related to output of gate g_k)

resources:
two resources (0_k and 1_k) for every gate g_k

latencies:
$\ell_{1_k}(x) = \mu(d)^k x^d$ and $\ell_{0_k}(x) = 3^{d/2}\mu(d)^k x^d$

strategies: every player has a *zero* and a *one* strategy, consisting of all 0- (1-) resources of direct successors (including own resource)



Properties (Gap Introduction)

For any $\alpha \in [1, 3^{d/2})$ it holds that in any α-PNE the players emulate the computation of the circuit and any α-PNE is an exact PNE.

Additional Results

The following problems are NP-hard, for polynomial congestion games of degree $d \geq 1$, for all $\alpha \in [1, 3^{d/2})$ and all $z > 0$:

- Does there exist an α-PNE in which a certain subset of players are playing a specific strategy profile?
- Does there exist an α-PNE in which a certain resource is used by at least one player?
- Does there exist an α-PNE in which a certain player has cost at most z?

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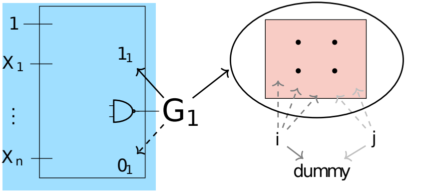
Combination of Gadgets

Combine *hardness (H)* and any *nonexistence (N)* gadget:

- scale weights and latencies in N to be smaller than values in H
- new dummy strategy for every player in N playing a new resource
- adjust one strategy of output player G_1 in H by adding resources from N

G_1 acts as mediator between the two gadgets



hardness gadget nonexistence gadget

Properties

- satisfying assignment $\rightarrow$ players in N play dummy strategy $\rightarrow$ α-PNE
- no satisfying assignment $\rightarrow$ players in N play in original game N $\rightarrow$ no α-PNE

Future Directions

- close nonexistence gaps
- complexity of finding α-PNE?

  ($d^{O(d)}$*-PNE* computable in poly. time

  exact PNE PLS-complete)

References

C. Hansknecht, M. Klimm, and A. Skopalik, Approximate Pure Nash Equilibria in Weighted Congestion Games (APPROX/RANDOM 2014)

I. Caragiannis and A. Fanelli, On Approximate Pure Nash Equilibria in Weighted Congestion Games with Polynomial Latencies (ICALP 2019)

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