Lecture: Approximation Algorithms

Jannik Matuschke

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Max Sat

Input: variables $x_1, \dots, x_n$, disjunctive clauses $C_1, \dots, C_m$, weights $w_1, \dots, w_m \in \mathbb{R}_+$

Task: find a truth assignment maximizing $\sum_{j : C_j \text{ is satisfied}} w_j$

$\underbrace{x_1 \vee \neg x_2 \vee x_3}_{C_1}$ ✓ $w_1 = 2$

$\underbrace{\neg x_1 \vee x_3}_{C_2}$ ✗ $w_2 = 3$

$\underbrace{\neg x_3}_{C_3}$ ✓ $w_3 = 1$

$\underbrace{x_2 \vee x_3 \vee x_4}_{C_4}$ ✓ $w_4 = 2$

$\underbrace{x_2 \vee \neg x_4}_{C_5}$ ✓ $w_5 = 1$

assignment: weight: 6

$x_1 = \texttt{true}$ $x_2 = \texttt{true}$ $x_3 = \texttt{false}$ $x_4 = \texttt{true}$

Previously ...

Algorithm (Random sampling):
For each i, set $x_i = \text{TRUE}$ with proability $1/2$ (independently).

Analysis: $\Pr[C_j \text{ satisfied}] = 1 - (1/2)^{|C_j|} \geq 1/2$

- Random sampling is a randomized $\frac{1}{2}$-approximation.
- Algorithm can be derandomized (Method of Conditional Expectations).

LP rounding

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{m} w_j z_j \\
\text{s.t.} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j & \forall\, j \in [m] \\
& y_i \in \{0, 1\} & \forall\, i \in [n] \\
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Algorithm 1:

1. Compute optimal LP solution (y^*, z^*).
2. For each $i \in [n]$, set x_i to `true` with probability y_i^*.

LP rounding

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\max & \displaystyle\sum_{j=1}^{m} w_j z_j & & \\
\text{s.t.} & \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j & & \forall\, j \in [m] \\
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Algorithm 1:

1. Compute optimal LP solution (y^*, z^*).
2. For each $i \in [n]$, set x_i to `true` with probability y_i^*.

Theorem 8.1

Algorithm 1 is a $(1 - 1/e)$-approximation algorithm for MAX SAT.

Choosing the better of two solutions

Let C_j be a clause of length k. From previous analysis:

- $\Pr[C_j \text{ sat. in random sampling}] \;\geq\; 1 - (1/2)^k$
- $\Pr[C_j \text{ sat. in randomized rounding}] \;\geq\; \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_j^*$

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Idea: Run both algorithms and take the better solution.

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Idea: Run both algorithms and take the better solution.

Analysis: Run either algorithm with probability $1/2$.
Then clause C_j is satisfied with probability at least

$$\frac{1}{2}\left(1 - 2^{-k}\right) z_j^* + \frac{1}{2}\left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_j^* \ \geq \ \frac{3}{4} z_j^*.$$

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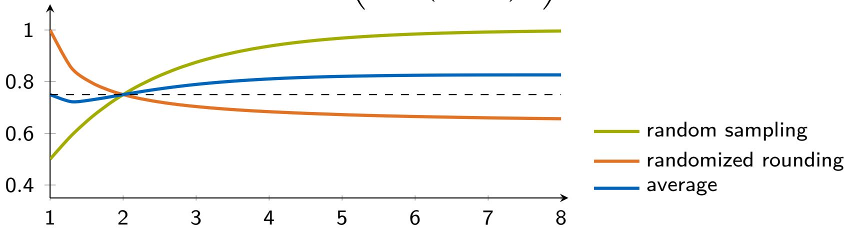
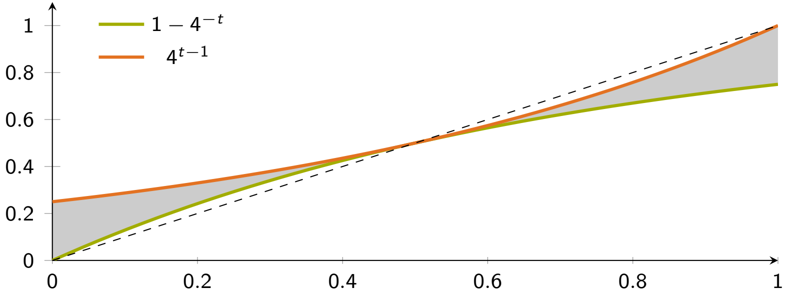


Non-linear randomized rounding

Let $f : [0,1] \to [0,1]$ be a function with

$$1 - 4^{-t} \leq f(t) \leq 4^{t-1} \qquad \text{for all } t \in [0,1].$$



Algorithm 2

1. Compute optimal LP solution (y^*, z^*).
2. For each $i \in [n]$, set x_i to `true` with probability $f(y_i^*)$.

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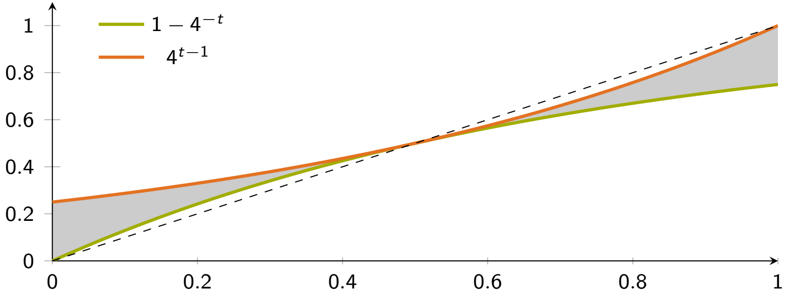

Algorithm 2

1. Compute optimal LP solution (y^*, z^*).
2. For each $i \in [n]$, set x_i to `true` with probability $f(y_i^*)$.

Theorem 8.2

Algorithm 2 is a randomized 3/4-approximation for MAX SAT.

Integrality gap

$$
\begin{aligned}
Z^* := \max \quad & \sum_{j=1}^{m} w_j z_j \\
\text{s.t.} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \ \geq \ z_j && \forall\, j \in [m] \\
& 0 \ \leq \ y_i \ \leq \ 1 && \forall\, i \in [n] \\
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We have analyzed two algorithms with ALG $\geq \frac{3}{4} Z^*$.

Can we do better using this LP?

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\end{aligned}
$$

We have analyzed two algorithms with ALG $\geq \frac{3}{4} Z^*$.
Can we do better using this LP? No.

Conisder this instance:

$x_1 \vee x_2 \qquad x_1 \vee \neg x_2 \qquad \neg x_1 \vee x_2 \qquad \neg x_1 \vee \neg x_2 \qquad\qquad w \equiv 1$

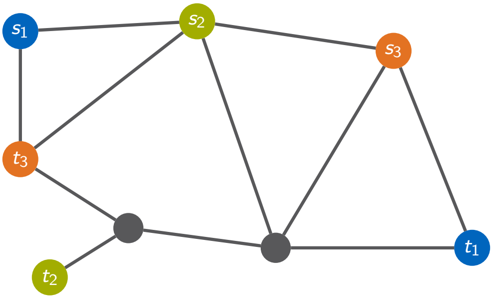
$$\text{OPT} = 3 \qquad Z^* = 4 \ (y_i = 1/2 \text{ for all } i)$$

Chernoff Bounds:
Integer Multicommodity Flows

Integer Multicommodity Flow

Input: graph $G = (V, E)$, k terminal pairs $s_i, t_i \in V$

Task: find set an s_i-t_i-path P_i for each $i \in [k]$, minimizing $\max_{e \in E} |\{i \in [k] : e \in P_i\}|$



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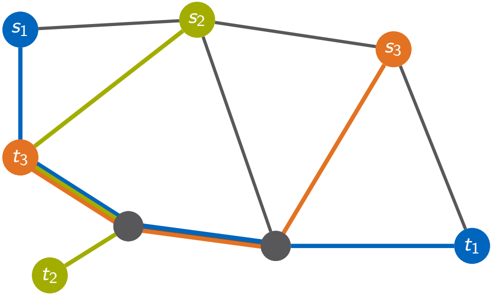

LP relaxation

$\mathcal{P}_i := \{P \subseteq E \,:\, P \text{ is } s_i\text{-}t_i\text{-path}\}$ $\qquad \mathcal{P} := \bigcup_{i\in[k]} \mathcal{P}_i$

$$
\begin{aligned}
\min \quad & W \\
\text{s.t.} \quad & \sum_{P\in\mathcal{P}_i} x_P = 1 && \forall\, i \in [k] \\
& \sum_{P\in\mathcal{P}:e\in P} x_P \leq W && \forall\, e \in E \\
& x_P \in \{0,1\} && \forall\, P \in \mathcal{P}
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Algorithm:

1. Compute optimal LP solution (x^*, W^*).
2. For each i, let $P_i = P \in \mathcal{P}_i$ with probability x_P^*.

Define random variable $Y_e := |\{i \,:\, e \in P_i\}|$. Then

$$\mathbb{E}[Y_e] = \sum_{i\in[k]} \sum_{P\in\mathcal{P}_i:e\in P} \Pr[P_i = P] = \sum_{i\in[k]} \sum_{P\in\mathcal{P}_i:e\in P} x_P^* \le W^*.$$

$$\mathbb{E}[\mathsf{ALG}] = \mathbb{E}[\max_{e \in E} Y_e]$$

Know: $\mathbb{E}[Y_e] \le W^* \le \mathsf{OPT}$

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Caution: $\mathbb{E}[\max_{e \in E} Y_e] \neq \max_{e \in E} \mathbb{E}[Y_e]$

Chernoff bound

Theorem 8.3

Let $X_1, \ldots, X_k$ be independent random variables in $\{0, 1\}$ and $U \geq \mathbb{E}[\sum_{i=1}^{k} X_i]$. Then for $0 \leq \delta \leq 1$:

$$\Pr\left[\sum_{i=1}^{k} X_i \geq (1+\delta)U\right] \leq \exp\left(-\frac{U\delta^2}{3}\right).$$

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Proof. Use Markov's inequality; see Williamson & Shmoys Section 5.10.

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Apply to Randomized Rounding for IMF:

Random variable $X_e^i = \begin{cases} 1 & \text{if path } P_i \text{ contains } e \\ 0 & \text{otherwise} \end{cases}$

Then $Y_e = \sum_{i=1}^k X_e^i$ with $\mathbb{E}[Y_e] \leq W^*$.

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With $\delta = 1$ and $U = c \ln(m) W^*$:

$$\Pr[Y_e \geq 2c\ln(m)W^*] \;\leq\; \exp\left(-\frac{c}{3}\ln(m)W^*\right) \;\leq\; m^{-c/3}$$

Analysis

Theorem 8.4

$\text{ALG} < 2c \ln(m) W^*$ with probability at least $1 - m^{1-c/3}$.

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$$\Pr[\text{ALG} \geq 2c\ln(m)W^*] \;=\; \Pr[\exists\, e \in E:\; Y_e \geq 2c\ln(m)W^*]$$

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$$\begin{aligned}\Pr[\text{ALG} \geq 2c\ln(m)W^*] &= \Pr[\exists\, e \in E:\ Y_e \geq 2c\ln(m)W^*] \\ &\underset{\bigcup\text{-bound}}{\leq} \sum_{e \in E} \Pr[Y_e \geq 2c\ln(m)W^*]\end{aligned}$$

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Remark: If $W^* \geq c\ln(m)$, we can get a better approximation guarantee; see Theorem 5.29 in Williamson & Shmoys.