MAX SATISFIABILITY

Analysis of Random Sampling

$$\Pr[C_j \text{ is satisfied}] = 1 - \Pr[\text{all literals of } C_j \text{ are false}]$$
$$= 1 - \left(\tfrac{1}{2}\right)^{|C_j|} \geq \tfrac{1}{2}$$

$$\mathbb{E}[ALG] = \sum_{j=1}^{m} w_j \cdot \Pr[C_j \text{ is satisfied}] \geq \tfrac{1}{2} \sum_{j=1}^{m} w_j \geq \tfrac{1}{2} OPT \quad \square$$

Derandomization (Method of Conditional Expectations)

Define event $B_i := (X_1 = b_1 \wedge \cdots \wedge X_i = b_i)$.

Show by induction: $\mathbb{E}[W \mid B_i] \geq \mathbb{E}[W]$ for all $i \in [n]$

For $i=1$: $$\mathbb{E}[W] = \tfrac{1}{2}\mathbb{E}[W \mid X_1 = 0] + \tfrac{1}{2}\mathbb{E}[W \mid X_1 = 1]$$
$$\underset{\text{choice of } b_1}{\leq} \mathbb{E}[W \mid X_1 = b_1]$$

For $i > 1$: $$\mathbb{E}[W] \leq \mathbb{E}[W \mid B_{i-1}]$$
$$= \tfrac{1}{2}\underbrace{\mathbb{E}[W \mid B_{i-1} \wedge X_i = 0]}_{W_0} + \tfrac{1}{2}\underbrace{\mathbb{E}[W \mid B_{i-1} \wedge X_i = 1]}_{W_1}$$
$$\underset{\text{choice of } b_i}{\leq} \mathbb{E}[W \mid B_i]$$

$$ALG = \mathbb{E}[W \mid B_n] \geq \mathbb{E}[W] \geq \frac{OPT}{2}$$

How to compute W_0, W_1?

$$\Pr[C_j \text{ is satisfied} \mid X_1 = b_1 \wedge \cdots \wedge X_k = b_k] = \begin{cases} 1 & \text{if } x_i \in C_j \text{ for some } i \text{ with } b_i = 1 \\ 1 & \text{if } \neg x_i \in C_j \text{ for some } i \text{ with } b_i = 0 \\ 1 - \left(\tfrac{1}{2}\right)^{k'} & \text{otherwise} \end{cases}$$

where $k' = |C_j \setminus \{x_i, \neg x_i : i \in [k]\}|$