Analysis of randomized rounding

$$\Pr[e \text{ not covered}] = \prod_{S: e\in S} (1-x(S))^{c\cdot \ln(n)} \underset{\substack{\uparrow \\ 1-x\leq \exp(-x)}}{\leq} \prod_{S:e\in S} \exp(-x(S))^{c\cdot \ln(n)}$$

$$= \exp\Big(-\sum_{S:e\in S} x(S)\, c\cdot \ln(n)\Big) \leq \overset{\exp(-c\cdot\ln(n))}{\cancel{\exp(-1)}} = \frac{1}{n^c}$$

Proof of Theorem 2.1

Define random variables $X(S) = \begin{cases} 1 & \text{if } S\in \mathcal{F}' \\ 0 & \text{otherwise} \end{cases}$ and

$X_i(S) = \begin{cases} 1 & \text{if ith coin for S shows heads} \\ 0 & \text{otherwise} \end{cases}$ for $S\in\mathcal{F}$, $i\in[c\ln(n)]$.

Note that $X(S) = \min\{\sum_{i=1}^{c\ln(n)} X_i(S), 1\}$ and therefore

$$\mathbb{E}[X(S)] \leq \mathbb{E}\Big[\sum_{i=1}^{c\ln(n)} X_i(S)\Big] = c\cdot \ln(n)\, x(S).$$

This implies:

$$\mathbb{E}\Big[\sum_{S\in\mathcal{F}'} w(S)\Big] = \mathbb{E}\Big[\sum_{S\in\mathcal{F}} w(S) X(S)\Big] \leq c\cdot \ln(n)\cdot \underbrace{\sum_{S\in\mathcal{F}} w(S)\, x(S)}_{=Z^*}.$$

Therefore:

$$\mathbb{E}\Big[\sum_{S\in\mathcal{F}'} w(S) \,\Big|\, \mathcal{F}' \text{ is a set cover}\Big] \leq \frac{\mathbb{E}[\sum_{S\in\mathcal{F}'} w(S)]}{\Pr[\mathcal{F}' \text{ is a set cover}]} \leq \frac{c\cdot\ln(n)\, Z^*}{1-\frac{1}{n^{c-1}}} \leq 2c\ln(n) Z^* \quad \square$$

for $n^{c-1} > 2$

Conditional expectations

random variable $X\geq 0$, event A

$$\mathbb{E}[X] = \Pr[A]\cdot \mathbb{E}[X|A] + \Pr[\neg A]\, \mathbb{E}[X|\neg A]$$

$$\mathbb{E}[X|A] = \frac{\mathbb{E}[X] - \Pr[\neg A]\cdot \mathbb{E}[X|\neg A]}{\Pr[A]} \leq \frac{\mathbb{E}[X]}{\Pr[A]}$$

The Traveling Salesman Problem (TSP)

Hamiltonian cycle (HC): spanning, connected subgraph every vertex has degree 2 (i.e., cycle containing all vertices)

Spanning tree: spanning connected subgraph, no cycles

Euler tour: closed walk that contains every edge exactly once
(H contains Euler tour iff H is connected and every vertex has even degree)

Analysis of Double-Tree: $d(C) \underset{\substack{\uparrow \\ \text{metric distances} \\ \text{shortcutting does} \\ \text{not increase cost}}}{\leq} d(C') \leq 2\,d(T) \underset{\substack{\uparrow \\ \text{tree lower bound}}}{\leq} 2 \cdot \mathrm{OPT}$ $\square$

Analysis of Christofides: $d(C) \leq d(C') \leq d(T) + d(M) \overset{\substack{\text{tree + matching} \\ \text{lower bound} \\ \downarrow}}{\leq} \frac{3}{2}\,\mathrm{OPT}$ $\square$